Differential Equations Part 1

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Abstract

We present a general method of solving a class of first order linear and nonlinear differential equations. We present general tests which can be run on the differential equations in question. If the equations satisfy the tests, the solutions to the differential equations can be written out directly. The analysis can be extended to a large class of differential equations, and can already tackle the general first order linear differential equation, the Bernoulli equation, separable differential equations, and a few differential equations we may be the first to solve.

1 Preliminaries

We will be considering differential equations solely of the form:

$$(a_1(x)b_1(y) + a_2(x)b_2(y) + ... + a_n(x)b_n(y))' = f(x)$$

where the conditions on $a_i(x)$, and $b_i(y)$ vary depending on the results, but generally include continuity and differentiability. Also, we assume $y(x)$ to be defined on some open region of $\mathbb{R}$.

Definition 1 *Let*

$$g(x,y)_x$$

denote the partial derivative of the differentiable function g(x,y) with respect to x, and let

$$g(x,y)_y$$

denote the partial derivative of g(x,y) with respect to y. We take y to be y(x), a function of x.

Definition 2 *Let*

$$f'(y)$$

be the partial derivative of f(y) with respect to y. Similarly, let

$$f'(x)$$

denote the partial derivative of f(x) with respect to x.

We begin by considering:

$$a_0(x)b_0(y) + a_1(x)b_1(y) + a_2(x)b_2(y) + a_3(x)b_3(y) + ... + a_n(x)b_n(y) = 0$$

Where we place greater restrictions on the functions $a_i(x)$ and $b_i(y)$ as n increases. These restrictions are not necessarily inherent in the problem but are placed to make our conclusions arise with greater ease.
For $n = 1$,

Theorem 1 *If*

$$a_0(x)b_0(y) + a_1(x)b_1(y) = 0$$

With

$$a_0(x)b_0(y) \neq 0 \text{ and } a_1(x)b_1(y) \neq 0$$

then

$$a_0(x) = ka_1(x), \text{ and } b_0(y) = -\frac{1}{k}b_1(y)$$

where

$$k \neq 0 \text{ is a constant.}$$

Proof: *Notice, since* $a_0(x)b_0(y) \neq 0$, *and* $a_1(x)b_1(y) \neq 0$, $a_1(x) \neq 0$, *and* $b_0(y) \neq 0$. *Therefore,*

$$a_0(x)b_0(y) + a_1(x)b_1(y) = 0 \tag{1}$$
$$a_0(x)b_0(y) = -a_1(x)b_1(y) \tag{2}$$
$$\frac{a_0(x)}{a_1(x)} = -\frac{b_1(y)}{b_0(y)} \tag{3}$$

And since the left hand side is independent of y, and the right hand side is independent of x,

$$\frac{a_0(x)}{a_1(x)} = k, \ -\frac{b_1(y)}{b_0(y)} = k \tag{4}$$
$$a_0(x) = ka_1(x), \ b_0(y) = \frac{-1}{k}b_1(y) \tag{5}$$

As was to be shown. Notice also, if on the other hand,

$$a_0(x)b_0(y) = 0 \text{ and } a_1(x)b_1(y) = 0$$

then both

$$\text{either } a_0(x) = 0 \text{ or } b_0(y) = 0$$

and

$$\text{either } a_1(x) = 0 \text{ or } b_1(y) = 0$$

must be true.

For n = 2,

Theorem 2 *If*

$$a_0(x)b_0(y) + a_1(x)b_1(y) + a_2(x)b_2(y) = 0$$

where $a_0(x), a_1(x),$ *and* $a_2(x)$ *are real functions of* x*, and* $b_0(y), b_1(y),$ *and* $b_2(y)$ *are real functions of* y*. then either:*
Case 1: Each term is zero. For example, this can happen if

$$a_1(x) = 0, a_2(x) = 0, a_3(x) = 0 \ or \tag{6}$$
$$a_1(x) = 0, a_2(x) = 0, b_3(y) = 0 \ , etc. \tag{7}$$

or
Case 2: One term is zero. This can happen if

$$a_1(x) = 0, a_2(x)b_2(y) = -a_3(x)b_3(y) \ or \tag{8}$$
$$b_1(y) = 0, a_2(x)b_2(y) = -a_3(x)b_3(y) \ , etc. \tag{9}$$

or
Case 3: All terms are nonzero
The proof of this goes without saying. If three terms add up to zero, then either one of the terms is zero, all of the terms are zero, or none of the terms are zero. The only thing left to do is characterize all the different ways the terms can be zero.

Notice, the terms of the second case all fit the form of Theorem 1. In this case, we know that the functions of x are proportional, and the functions of y are proportional.

Adding a third term in the sum of our equation has provided us with 8 uninteresting possibilities ((6) and (7)), 6 possibilities that reduce to the case of Theorem 1 ((8) and (9)), and one case that is brand new. Let us consider the brand new case.

Theorem 3 *If*

$$a_0(x)b_0(y) + a_1(x)b_1(y) + a_2(x)b_2(y) = 0$$

where $a_0(x), a_1(x),$ *and* $a_2(x)$ *are real functions of the real variable* x*,* $b_0(y), b_1(y),$ *and* $b_2(y)$ *are real functions of the real variable* y*,*

$$a_0(x)b_0(y), \ a_1(x)b_1(y), \ and \ a_2(x)b_2(y) \neq 0,$$

and either $a_0(x), a_1(x),$ *and* $a_2(x)$ *are differentiable with respect to* x*,*
or $b_0(y), b_1(y),$ *and* $b_2(y)$ *are differentiable with respect to* y*.*
then either

$$a_0(x) = k_1 a_1(x)$$

and

$$a_1(x) = k_2 a_2(x)$$

where k_1*,and* $k_2 \neq 0$ *are constants*
or

$$b_0(y) = k_3 b_1(y)$$

and

$$b_1(y) = k_4 b_2(y)$$

where k_3*,and* $k_4 \neq 0$ *are constants.*
Proof: *Without loss of generality, assume that the functions of* y *(*$b_0(y), b_1(y)$*, and* $b_2(y)$*) are differentiable. In this analysis, there is no distinction between the variable* x*, and* y*. Y need not be a function of* x*, so if instead the functions of* x *are differentiable, we could have replaced* x *with* y *and the analysis would be the same. Notice, for starters,* $b_2(y) \neq 0$*, so*

$$a_0(x)b_0(y) + a_1(x)b_1(y) + a_2(x)b_2(y) = 0$$
$$a_0(x)\frac{b_0(y)}{b_2(y)} + a_1(x)\frac{b_1(y)}{b_2(y)} + a_2(x) = 0$$

Now, take the partial derivative with respect to y*. This is the heart of the technique we use to reduce this theorem to previous ones.*

$$\left(a_0(x)\frac{b_0(y)}{b_2(y)} + a_1(x)\frac{b_1(y)}{b_2(y)} + a_2(x)\right)_y = 0$$
$$a_0(x)\left(\frac{b_0(y)}{b_2(y)}\right)' + a_1(x)\left(\frac{b_1(y)}{b_2(y)}\right)' = 0$$

Notice, neither $a_0(x)$ *nor* $a_1(x)$ *are zero. But this implies, by theorem 1, either*

$$\left(\frac{b_0(y)}{b_2(y)}\right)' = \left(\frac{b_1(y)}{b_2(y)}\right)' = 0,$$

or

$$a_0(x) = ka_1(x).$$

But if

$$\left(\frac{b_0(y)}{b_2(y)}\right)' = \left(\frac{b_1(y)}{b_2(y)}\right)' = 0,$$

then

$$b_0(y) = k_1 b_2(y)$$

and

$$b_1(y) = k_2 b_2(y)$$

by differentiability. And if

$$a_0(x) = ka_1(x),$$

then

$$a_0(x)b_0(y) + a_1(x)b_1(y) + a_2(x)b_2(y) = 0$$
$$a_0(x)\left(b_0(y) + \frac{1}{k}b_1(y)\right) + a_2(x)b_2(y) = 0$$

So by Theorem 1, we have

$$a_0(x) = k_2 a_2(x)$$

In both cases, we have shown that either the x *functions are proportional or the* y *functions are proportional. The constants of proportionality are nonzero simply because all of the terms are nonzero.*

In this way, we can imagine how one might attempt to fully classify the conditions the functions of x and the functions of y must satisfy as we add more terms. The only trick is to divide the sum by a nonzero term, and differentiate with respect to one of the variables. The problem then reduces to one of the previous theorem.

This brings us to the meat of the paper. When can the DE

$$\frac{d}{dx}y(x) = f_1(x)g_1(y) + f_2(x)g_2(y)$$

be written in the form

$$\frac{d}{dx}(a(x)b(y)) = c(x)$$

(whose solution is easily obtainable by integrating and isolating y)? Notice if the DE

$$\frac{d}{dx}y(x) = f_1(x)g_1(y) + f_2(x)g_2(y)$$

can be written in the form

$$\frac{d}{dx}(a(x)b(y)) = c(x)$$

then, isolating y gives us.

$$\int d(a(x)b(y)) = \int c(x)dx$$

$$a(x)b(y) = C(x) + k$$

$$b(y) = \frac{C(x) + k}{a(x)}$$

$$b(y) = Q(x) + k \cdot P(x)$$

This solution set has an arbitrary constant k. This set is a solution regardless of the value of k. Finally, $P(x) \neq 0$. Thus, the remarkable thing about the following analysis, is the orientation of the arbitrary constant in the solution set of a differential equation is what determines whether or not the tests we devise will solve the differential equations. In fact, the converse can be seen to hold as well. That is, if for all $k \in \mathbb{R}$,

$$b(y) = Q(x) + k \cdot P(x)$$

solves

$$\frac{d}{dx}y(x) = f_1(x)g_1(y) + f_2(x)g_2(y)$$

where $P(x)$ is non zero, then

$$\frac{d}{dx}y(x) = f_1(x)g_1(y) + f_2(x)g_2(y)$$

can be written in the form

$$\frac{d}{dx}(a(x)b(y)) = c(x).$$

However, before we explore this any further, we will take a small detour outlining the power of partial differential equations for describing functions. Many of the cases we consider in our later work will reduce

$$\frac{d}{dx}y(x) = f_1(x)g_1(y) + f_2(x)g_2(y)$$

to a separable differential equation. So before we proceed, we will outline how to use partial differential equations to identify a separable differential equation.

Corollary 1 *Separability*
A real value function of real variables x and y, who's log is differentiable in x and y, can be written as a product of functions of x and y if and only if it satisfies the partial differential equation:

$$ln(f(x,y))_{xy} = 0$$

Proof
Let $f(x,y) = a(x)b(y)$, then $ln(a(x)b(y)) = ln(a(x)) + ln(b(y))$ so $ln(a(x)b(y))_{xy} = ln(a(x))_{xy} + ln(b(y))_{xy} = 0$
Let $ln(f(x,y))_{xy} = 0$, then $ln(f(x,y))_x = a(x)$, and $ln(f(x,y)) = b(y) + \int a(x)$ so $f(x,y) = e^{b(y)}e^{\int a(x)}$

Example: Separable differential equation

Solve

$$y' = f(x)((xy)^2 + x^2 + y^2 + 1)$$

It may not look like it, but this is a separable differential equation. To see this, let us apply the test described in corollary 1.

$$\begin{aligned}
&ln(f(x)((xy)^2 + x^2 + y^2 + 1))_{xy} \\
&\qquad = ln(f(x))_{xy} + ln((xy)^2 + x^2 + y^2 + 1)_{xy} \\
&\qquad = \left(\frac{2xy^2 + 2x}{(xy)^2 + x^2 + y^2 + 1}\right)_y \\
&\qquad = \frac{(x^2y^2 + x^2 + y^2 + 1)(4xy) - (2x^2y + 2y^2)(2xy^2 + 2x)}{((xy)^2 + x^2 + y^2 + 1)^2} \\
&\qquad = \frac{4x^3y^3 + 4x^3y + 4xy^3 + 4xy - (4x^3y^3 + 4x^3y + 4xy^3 + 4xy)}{((xy)^2 + x^2 + y^2 + 1)^2} \\
&\qquad = 0
\end{aligned}$$

Our function passes the test, so it is separable, (except perhaps at the point where the denominator is zero). As this is just a single point, we aren't concerned. If anything, we can fill that point in after we derive our solution. So let's separate the function! Notice, it isn't hard to see that the quadratic term factors into $(x^2 + 1)(y^2 + 1)$. But we will show you how we can do this without assuming we can factor the quadratic function. If we could factor it, we would have known the function is separable from the get go. We want to separate

$$f(x)(x^2y^2 + x^2 + y^2 + 1)$$

Let us pick an x value such that $f(x) \neq 0$ and $x^2y^2 + x^2 + y^2 + 1 \neq 0$. Any real number for which $f(x) \neq 0$ will suffice. Call this x, x_0 and $f(x_0)$, f_0. Now our function is separable, so

$$f(x)(x^2y^2 + x^2 + y^2 + 1) = g(x)p(y)$$

for some functions $g(x)$ and $p(y)$. In particular,

$$f_0((x_0)^2y^2 + (x_0)^2 + y^2 + 1) = g(x_0)p(y)$$

and choosing $y_0 = 0$

$$f(x)(x^2 + 1) = g(x)p(0)$$

So,

$$p(0)g(x_0)g(x)p(y) = f(x)(x^2 + 1)f_0((x_0)^2y^2 + (x_0)^2 + y^2 + 1)$$

Now,

$$p(0)g(x_0) = f_0(x_0^2 + 1)$$

So

$$g(x)p(y) = \frac{f(x)(x^2 + 1)((x_0)^2y^2 + (x_0)^2 + y^2 + 1)}{(x_0^2 + 1)}$$

Thus, we have separated the variables. Let

$$g(x) = f(x)(x^2 + 1)$$

and

$$\begin{aligned} p(y) &= \frac{(x_0)^2y^2 + (x_0)^2 + y^2 + 1}{(x_0)^2 + 1} \\ &= \frac{y^2((x_0)^2 + 1) + (x_0)^2 + 1}{(x_0)^2 + 1} \\ &= y^2 + 1 \end{aligned}$$

This process works to separate any equation. Step 1, verify the equation is separable, and step 2, separate it by plugging in inputs that generate nonzero outputs. To finish our solution,

$$y' = g(x)p(y) \Rightarrow \int p(y)^{-1} = \int g(x)$$

So,

$$tan^{-1}(y) = \int (f(x)(x^2 + 1)) + c \Rightarrow y = tan\left(\int (f(x)(x^2 + 1)) + c\right)$$

In summary, not only do we have a test for separability of the function, but, if we show the function is separable, we can separate it with algebra that can be done autonomously.
To emphasize the utility of this sort of analysis, consider the wave equation.

$$U_{xx}(x, y) = c^2U_{yy}(x, y)$$

The general solution to this equation is

$$f(x+cy)+g(x-cy)$$

where f and g are doubly differentiable functions. It can be stated if a function $p(x,y)$ can be written as

$$f(x+cy)+g(x-cy)$$

for some doubly differentiable functions f, and g, then $P(x,y)$ will satisfy the wave equation, and the converse is also true. Thus, we can use the wave equation as a test which will determine the nature of a function of two variables.

Theorem 4 *A function $Q(x,y)$ which is doubly differentiable in x, and y, can be written as $f(x+cy)+g(x-cy)$ for some doubly differentiable functions f and g, and nonzero constant c, if and only if $Q(x,y)$ satisfies the wave equation.*
Proof *The proof of this can be found in any book on partial differential equations.*

Example: Wave equation

Verify that

$$sin(x)\cdot cos(y)$$

can be written in the form

$$f(x+y)+g(x-y)$$

for some doubly differentiable functions f and g and find f, and g.
Solution: First,

$$(sin(x)\cdot cos(y))_{xx}=-sin(x)\cdot cos(y)$$

and

$$(sin(x)\cdot cos(y))_{yy}=-sin(x)\cdot cos(y)$$

so if we let $c=1$, this function does satisfy the wave equation. The rest, like in the previous example, is just some clever algebra. For any wave equation, assuming

$$Q(x,y)=f(x+cy)+g(x-cy)$$

gives us

$$Q(x,y)_x=f'(x+cy)+g'(x-cy)$$

and

$$Q(x,y)_y=cf'(x+cy)-cg'(x-cy).$$

Combining these two statements, and, assuming $Q(x,y)$ to be defined at $y=0$, we get:

$$\begin{aligned}
f'(x+cy) &= \frac{1}{2}\left(Q(x,y)_x+\frac{1}{c}Q(x,y)_y\right)\\
g'(x-cy) &= \frac{1}{2}\left(Q(x,y)_x-\frac{1}{c}Q(x,y)_y\right)\\
f'(x) &= \frac{1}{2}\left(Q_x(x,0)+\frac{1}{c}Q_y(x,0)\right)\\
g'(x) &= \frac{1}{2}\left(Q_x(x,0)-\frac{1}{c}Q_y(x,0)\right)
\end{aligned}$$

So in our case,

$$f'(x) = \frac{1}{2}\left(cos(x)cos(0) + -sin(x)sin(0)\right) = \frac{1}{2}cos(x)$$

and

$$g'(x) = \frac{1}{2}\left(cos(x)cos(0) - -sin(x)sin(0)\right) = \frac{1}{2}cos(x)$$

Therefor,

$$g(x) = \frac{1}{2}sin(x) + c,$$

and

$$f(x) = \frac{1}{2}sin(x) + c_2.$$

So,

$$sin(x)cos(y) = \frac{1}{2}\left(sin(x+y) + sin(x-y)\right) + c + c_2 = \frac{1}{2}\left(sin(x+y) + sin(x-y)\right) + c_3.$$

Plugging 0 in for x gives us $c_3 = 0$ so

$$sin(x)cos(y) = \frac{1}{2}(sin(x+y) + sin(x-y)).$$

2 Main Results

Theorem 5 *First Major Result*

A first order differential equation:

$$y' = a_1(x)b_1(y) + a_2(y)b_2(y) \ (6)$$

with $b_1(y)$ and $b_2(y)$ doubly differentiable, can be written in the form

$$(a(x)b(y))' = c(x) \ (7)$$

for some functions $a(x)$, $b(y)$, and $c(x)$ where $a(x)$, and $b'(y)$ are non zero, only under one of the following 3 conditions.

1) The equation reduces to a separable differential equation

2) $b_1(y)$, and $b_2(y)$, satisfy

either: $b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = k$

or, $b_2(y)\left(\frac{b_1(y)}{b_2(y)}\right)' = k$

for some non-zero constant k.

3) Or, the functions $b_1(y)$, and $b_2(y)$ satisfy

$k_1b_2(y) - k_2b_1(y) = b_1'(y)b_2(y) - b_1(y)b_2'(y)$

for some nonzero constants k_1, k_2.

Proof: *Assume (6) is satisfied by (7). Then (7) can be rearranged into:*

$$a'(x)b(y) + a(x)b'(y)y' = c(x)$$
$$a(x)b'(y)y' = c(x) - a'(x)b(y)$$

And (6) can be rearranged into:

$$a(x)b'(y)y' = (a_1(x)b_1(y) + a_2(x)b_2(y))a(x)b'(y)$$
$$a(x)b'(y)y' = a(x)a_1(x)b'(y)b_1(y) + a(x)a_2(x)b'(y)b_2(y)$$

Setting these two equal to each other, and taking the partial derivative with respect to y gives:

$$\begin{aligned} c(x) - a'(x)b(y) &= a(x)a_1(x)b'(y)b_1(y) + a(x)a_2(x)b'(y)b_2(y) \\ -a'(x)b'(y) &= a(x)a_1(x)(b'(y)b_1(y))' + a(x)a_2(x)(b'(y)b_2(y))' \\ 0 &= a'(x)b'(y) + a(x)a_1(x)(b'(y)b_1(y))' + a(x)a_2(x)(b'(y)b_2(y))' \quad (8) \end{aligned}$$

If we refer back to the classification of the three term function product equation, and theorem 2, we can list all of the possible ways (8) can be satisfied:

Case 1: Everything is zero

$$\begin{aligned} a'(x) &= 0, a(x)a_1(x) = 0, a(x)a_2(x) = 0 \\ a'(x) &= 0, a(x)a_1(x) = 0, (b'(y)b_2(y))' = 0 \\ a'(x) &= 0, (b'(y)b_1(y))' = 0, a(x)a_2(x) = 0 \\ a'(x) &= 0, (b'(y)b_1(y))' = 0, (b'(y)b_2(y))' = 0 \\ b'(y) &= 0, a(x)a_1(x) = 0, a(x)a_2(x) = 0 \\ b'(y) &= 0, a(x)a_1(x) = 0, (b'(y)b_2(y))' = 0 \\ b'(y) &= 0, (b'(y)b_1(y))' = 0, a(x)a_2(x) = 0 \\ b'(y) &= 0, (b'(y)b_1(y))' = 0, (b'(y)b_2(y))' = 0 \end{aligned}$$

We have assumed $b'(y)$ is non zero, therefor, we can throw out the later half of these cases. Also, since we have assumed $a(x) \neq 0$ the remaining four cases must have $a(x) = k$ for some nonzero constant k. It follows $a(x)a_1(x) = 0$ implies $a_1(x) = 0$ this reduces (6) to a separable differential equation, similarly, $a(x)a_2(x) = 0$ implies $a_2(x) = 0$. Both of these, at best reduce equation (6) to separable and at worst reduce it to the trivial $y' = 0$ equation. It follows, the only case left to consider is the case

$$a'(x) = 0,\ (b'(y)b_1(y))' = 0,\ (b'(y)b_2(y))' = 0$$
$$a'(x) = 0,\ b'(y)b_1(y) = c1,\ b'(y)b_2(y) = c2$$

This implies both $b_1(y)$ and $b_2(y)$ are proportional to $\frac{1}{b'(y)}$ so $b_1(y)$ and $b_2(y)$ are proportional to one another. Thus, equation (6) is separable. For case 1, where all terms of equation (8) are zero, equation (6) is separable.

Case 2: One term is zero

$$
\begin{aligned}
a'(x) = 0, a(x)a_1(x)(b'(y)b_1(y))' &= -a(x)a_2(x)(b'(y)b_2(y))' \\
b'(y) = 0, a(x)a_1(x)(b'(y)b_1(y))' &= -a(x)a_2(x)(b'(y)b_2(y))' \\
a(x)a_1(x) = 0, a'(x)b'(y) &= -a(x)a_2(x)(b'(y)b_2(y))' \\
(b'(y)b_1(y))' = 0, a'(x)b'(y) &= -a(x)a_2(x)(b'(y)b_2(y))' \\
a(x)a_2(x) = 0, a'(x)b'(y) &= -a(x)a_1(x)(b'(y)b_1(y))' \\
(b'(y)b_2(y))' = 0, a'(x)b'(y) &= -a(x)a_1(x)(b'(y)b_1(y))'
\end{aligned}
$$

Again, we can assume $b'(y) \neq 0$ this implies we can throw out

$$b'(y) = 0, a(x)a_1(x)(b'(y)b_1(y))' = -a(x)a_2(x)(b'(y)b_2(y))'$$

Again, $a'(x) = 0$, implies $a(x) = k \neq 0$. so

$$a'(x) = 0, a(x)a_1(x)(b'(y)b_1(y))' = -a(x)a_2(x)(b'(y)b_2(y))'$$

implies

$$ka_1(x) = -ka_2(x)\frac{(b'(y)b_2(y))'}{(b'(y)b_1(y))'} = -ka_2(x)k_2$$

Since $\frac{(b'(y)b_2(y))'}{(b'(y)b_1(y))'}$ is the only term dependent on y, it must be constant. So $a_1(x)$ is proportional to $a_2(x)$ and equation (6) is separable. Also, $a(x)a_1(x) = 0$ implies $a_1(x) = 0$ and $a(x)a_2(x) = 0$ implies $a_2(x) = 0$. Both of these cases reduce equation (6) to a separable equation. The only two possibilities for this case remaining are

$$
\begin{aligned}
&(b'(y)b_1(y))' = 0, a'(x)b'(y) = -a(x)a_2(x)(b'(y)b_2(y))' \\
&(b'(y)b_2(y))' = 0, a'(x)b'(y) = -a(x)a_1(x)(b'(y)b_1(y))' \\
&(b'(y)b_1(y))' = 0 \Rightarrow b'(y)b_1(y) = k \neq 0 \Rightarrow b'(y) = \frac{k}{b_1(y)} \text{ and} \\
&\quad a'(x)b'(y) = -a(x)a_2(x)(b'(y)b_2(y))' \Rightarrow b'(y) = c(b'(y)b_2(y))' \; c \neq 0 \\
&\quad \Rightarrow \frac{k}{b_1(y)} = c\left(\frac{k}{b_1(y)}b_2(y)\right)' \Rightarrow \frac{1}{c} = \left(\frac{b_2(y)}{b_1(y)}\right)' b_1(y)
\end{aligned}
$$

A similar argument shows,

$$(b'(y)b_2(y))' = 0, a'(x)b'(y) = -a(x)a_1(x)(b'(y)b_1(y))'$$

Implies

$$\frac{1}{c} = \left(\frac{b_1(y)}{b_2(y)}\right)' b_2(y)$$

Thus, the case where only one of the terms was zero has yielded separable equations, and equations that satisfy the second test of the theorem. The last case to consider is the case where none of the terms of equation (6) are zero. If we recall from Theorem 3, this implies either the x dependent functions are proportional, or the y dependent functions are proportional. Either,

$$a'(x) = k_1 a(x)a_1(x), a'(x) = k_2 a(x)a_2(x)$$

or,

$$b'(y) = k_3(b'(y)b_1(y))', b'(y) = k_4(b'(y)b_2(y))' \; k_3, k_4 \neq 0$$

If the $a(x)$ terms are proportional, equation (16) reduces to a separable differential equation. working with these conditions gives us

$$\begin{aligned}
b(y) &= k_3 b'(y)b_1(y) + k_5 \\
b(y) &= k_4 b'(y)b_2(y) + k_6 \\
k_4 b'(y)b_2(y) + k_6 &= k_3 b'(y)b_1(y) + k_5 \\
(k_4 b_2(y) - k_3 b_1(y))b'(y) &= k_5 - k_6
\end{aligned}$$

If $k_5 - k6 = 0$, then $k_4 b_2(y) - k_3 b_2(y) = 0$, since $b'(y) \neq 0$. This reduces equation (6) to a separable equation. Otherwise,

$$b'(y) = \frac{k_5 - k_6}{k_4 b_2(y) - k_3 b_1(y)}$$

And

$$b(y) = k_3 b'(y)b_1(y) + k_5 \Rightarrow b'(y) = k_3(b'(y)b_1(y))'$$

$$\Rightarrow \frac{k_5 - k_6}{k_4 b_2(y) - k_3 b_1(y)} = k_3 \left(\frac{k_5 - k_6}{k_4 b_2(y) - k_3 b_1(y)} b_1(y) \right)'$$

The rest, like most everything prior, is just a matter of algebra.

$$\begin{aligned}
\frac{k_5 - k_6}{k_4 b_2(y) - k_3 b_1(y)} &= k_3 \left(\frac{k_5 - k_6}{k_4 b_2(y) - k_3 b_1(y)} b_1(y) \right)' \\
\frac{1}{k_3} &= \left(\frac{b_1(y)}{k_4 b_2(y) - k_3 b_1(y)} \right)' (k_4 b_2(y) - k_3 b_1(y)) \\
\frac{1}{k_3} &= \left(\frac{b_1(y)}{\frac{k_4}{k_3} b_2(y) - b_1(y)} \right)' \left(\frac{k_4}{k_3} b_2(y) - b_1(y) \right) \\
\frac{1}{k_3} &= \left(\frac{b_1'(y)\left(\frac{k_4}{k_3} b_2(y) - b_1(y)\right) - b_1(y)\left(\frac{k_4}{k_3} b_2'(y) - b_1'(y)\right)}{\left(\frac{k_4}{k_3} b_2(y) - b_1(y)\right)^2} \right) \left(\frac{k_4}{k_3} b_2(y) - b_1(y) \right) \\
\frac{1}{k_3} &= \left(\frac{\frac{k_4}{k_3} b_1'(y)b_2(y) - \frac{k_4}{k_3} b_1(y)b_2'(y)}{\frac{k_4}{k_3} b_2(y) - b_1(y)} \right) \\
\frac{1}{k_3} &= \left(\frac{b_1'(y)b_2(y) - b_1(y)b_2'(y)}{b_2(y) - \frac{k_3}{k_4} b_1(y)} \right) \\
\frac{1}{k_3}(b_2(y) - \frac{k_3}{k_4} b_1(y)) &= b_1'(y)b_2(y) - b_1(y)b_2'(y) \\
\frac{1}{k_3} b_2(y) - \frac{1}{k_4} b_1(y) &= b_1'(y)b_2(y) - b_1(y)b_2'(y)
\end{aligned}$$

The only thing worth mentioning is we could eliminate $k_5 - k_6$ because it is nonzero. This factor does not change the test, so without loss of generality, we can choose $k_5 = 1$ and $k_6 = 0$. This completes our proof.

And now, for the fun part!

Theorem 6 *First Test*
Let

$$y'(x) = a_1(x)b_1(y) + a_2(x)b_2(y)$$

if

$$b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = k \neq 0$$

or if

$$b_2(y)\left(\frac{b_1(y)}{b_2(y)}\right)' = k \neq 0$$

Then

$$e^{-k\int a_2(x)}\frac{b_2(y)}{b_1(y)} = k\int a_1(x)e^{-k\int a_2(x)}$$

or

$$e^{-k\int a_1(x)}\frac{b_1(y)}{b_2(y)} = k\int a_2(x)e^{-k\int a_1(x)}$$

solves the differential equation respectively.
Proof:
Suppose

$$b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = k$$

$$\begin{aligned}
y'(x) &= a_1(x)b_1(y) + a_2(x)b_2(y)\\
-a_2(x)b_2(y) + y'(x) &= a_1(x)b_1(y)\\
-ka_2(x)b_2(y) + ky'(y) &= ka_1(x)b_1(y)\\
-ka_2(x)b_2(y) + b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' y'(x) &= ka_1(x)b_1(y)\\
-ka_2(x)\frac{b_2(y)}{b1(y)} + \left(\frac{b_2(y)}{b_1(y)}\right)' y'(x) &= ka_1(x)\\
e^{-k\int a_2(x)}(-ka_2(x))\frac{b_2(y)}{b1(y)} + e^{-k\int a_2(x)}\left(\frac{b_2(y)}{b_1(y)}\right)' y'(x) &= ka_1(x)e^{-k\int a_2(x)}\\
\left(e^{-k\int a_2(x)}\frac{b_2(y)}{b1(y)}\right)' &= ka_1(x)e^{-k\int a_2(x)}\\
e^{-k\int a_2(x)}\frac{b_2(y)}{b1(y)} &= k\int a_1(x)e^{-k\int a_2(x)}
\end{aligned}$$

A similar argument covers the other case.

Notice, our solution is only technically valid over the regions where $b_1(y)$ (in case 1), and $b_2(y)$ (in case 2) is non zero. Clearly, if $b_1(y)$ or $b_2(y)$ are identically zero, we reduce to the simple case of a separable differential equation. If $k = 0$ we also reduce to the trivial separable differential equation case. Also, this proof is not very instructive, however, if you refer back to the previous theorem, you will recall this first test ensures the differential equation is of the form

$$(a(x)b(y))' = c(x)$$

with $a(x), b(y)$, and $c(x)$ suggested by theorem 4.

Theorem 7 *The second test*
Let

$$y' = a_1(x)b_1(y) + a_2(x)b_2(y)$$

Where $b_1(y), b_2(y)$ *are doubly differentiable. If*

$$\frac{1}{k_3}b_2(y) - \frac{1}{k_4}b_1(y) = b_1'(y)b_2(y) - b_1(y)b_2'(y)$$

Then

$$k_3 = \frac{\left(\frac{b_2(y)}{b_1(y)}\right)'}{\left(\frac{b_1'(y)b_2(y)}{b_1(y)}\right)' - b_2''(y)}$$

$$k_4 = \frac{\left(\frac{b_1(y)}{b_2(y)}\right)'}{\left(\frac{b_1(y)b_2'(y)}{b_2(y)}\right)' - b_1''(y)}$$

And

$$a(x)b(y) = \int c(x)$$

Solves the differential equation where,

$$a(x) = e^{-\int\left(\frac{a_1(x)}{k_3} + \frac{a_2(x)}{k_4}\right)}$$

$$b(y) = \frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)}$$

And

$$c(x) = -\frac{a_1(x)}{k_3}e^{-\int\left(\frac{a_1(x)}{k_3} + \frac{a_2(x)}{k_4}\right)}$$

proof:

$$\begin{aligned}
\frac{1}{k_3}b_2(y) - \frac{1}{k_4}b_1(y) &= b_1'(y)b_2(y) - b_1(y)b_2'(y) \\
\frac{1}{k_3}\frac{b_2(y)}{b_1(y)} - \frac{1}{k_4} &= \frac{b_1'(y)b_2(y)}{b_1(y)} - b_2'(y) \\
\frac{1}{k_3}\left(\frac{b_2(y)}{b_1(y)}\right)' &= \left(\frac{b_1'(y)b_2(y)}{b_1(y)}\right)' - b_2''(y) \\
k_3 &= \frac{\left(\frac{b_2(y)}{b_1(y)}\right)'}{\left(\frac{b_1'(y)b_2(y)}{b_1(y)}\right)' - b_2''(y)}
\end{aligned}$$

A similar argument gives us:

$$k_4 = \frac{\left(\frac{b_1(y)}{b_2(y)}\right)'}{\left(\frac{b_1(y)b_2'(y)}{b_2(y)}\right)' - b_1''(y)}$$

Notice, in the last line of this demonstration, We may be dividing by zero. However, since $b_2(y)$ *is not proportional to* $b_1(y)$, $\left(\frac{b_2(y)}{b_1(y)}\right)'$ *is not identically*

zero, $k_3 \neq 0$ so we are not dividing by a function that is identically zero. Thus we can evaluate our equation where it is non zero, and still obtain a value for k_3. A similar statement holds for k_4. Now for part 2 of the theorem.

$$\begin{aligned}
y' &= a_1(x)b_1(y) + a_2(x)b_2(y) \\
-a_2(x)b_2(y) + y' &= a_1(x)b_1(y) \\
-\frac{k_4}{k_3}a_1(x)b_2(y) - a_2(x)b_2(y) + y' &= a_1(x)b_1(y) - \frac{k_4}{k_3}a_1(x)b_2(y) \\
-\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right) k_4 b_2(y) + y' &= -\frac{1}{k_3}a_1(x)\left(k_4 b_2(y) - k_3 b_1(y)\right) \\
-\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right) \frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)} + \frac{y'}{k_4 b_2(y) - k_3 b_1(y)} &= -\frac{1}{k_3}a_1(x) \\
-\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right) \frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)} + \frac{k_4 k_3 \left(\frac{b_2(y)}{k_3} + \frac{b_1(y)}{k_4}\right) y'}{(k_4 b_2(y) - k_3 b_1(y))^2} &= -\frac{1}{k_3}a_1(x) \\
-\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right) \frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)} + \frac{k_4 k_3 (b_1'(y) b_2(y) - b_1(y) b_2'(y)) y'}{(k_4 b_2(y) - k_3 b_1(y))^2} &= -\frac{1}{k_3}a_1(x) \\
-\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right) \frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)} + \left(\frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)}\right)' y' &= -\frac{1}{k_3}a_1(x) \\
e^{-\int\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right)} \left(-\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right) \frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)} + \left(\frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)}\right)' y'\right) &= e^{-\int\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right)} \left(-\frac{1}{k_3}a_1(x)\right) \\
\left(e^{-\int\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right)} \left(\frac{k_4 b_2(y)}{k_4 b_2(y) - k_3 b_1(y)}\right)\right)' &= e^{-\int\left(\frac{1}{k_3}a_1(x) + \frac{1}{k_4}a_2(x)\right)} \left(-\frac{1}{k_3}a_1(x)\right) \\
(a(x)b(y))' &= c(x) \\
a(x)b(y) &= \int c(x)
\end{aligned}$$

Notice, in this part of the proof we also divide by a function which may be zero. However, this possibility was considered in theorem 4.

The next test is one that was derived assuming an extra term in the solution which is a function of y $((a(x)b(y) + c(y))' = d(x))$. This results in a sum of products of functions with four terms, which can be satisfied in a vast multitude of ways (refer to theorem 3 for the multitude of ways with 3 terms). we chose just a small handful of these and derived a corresponding test much like we had for the previous tests. This test does not appear to be satisfied for any nice differential equations so, we decided not to pursue this avenue further. Notice also, the conditions are now both imposed on the functions of x as well as the functions of y.

Theorem 8 *Another unrelated test*
Let

$$y'(x) = f_1(x)g_1(y) + f_2(x)g_2(y)$$

if $\left(\frac{f_1(x)}{f_2(x)}\right)' = kf_1(x)$ *and* $\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right)' = -\frac{k}{g_1(y)}$ *Then*

$$\int e(x) = a(x)b(y) + c(y)$$

Solves the differential equation where $e(x) = f_2(x)k_2 + \frac{(f_1(x))^2}{f_2(x)}$, $a(x) = \frac{f_1(x)}{f_2(x)}$, $b(y) = -\frac{1}{k}\left(\frac{g_2(y)}{g_1(y)} + \frac{g_1(y)}{g_2}k_2\right)$, *and* $c(y) = \int \frac{k_2}{g_2(y)}$

Proof:

$$
\begin{aligned}
y'(x) &= f_1(x)g_1(y) + f_2(x)g_2(y) \\
\left(\frac{1}{g_1(y)}\left(\frac{f_1(x)}{f_2(x)}\right) + k_2\frac{1}{g_2(y)}\right)y'(x) &= \left(\frac{1}{g_1(y)}\left(\frac{f_1(x)}{f_2(x)}\right) + k_2\frac{1}{g_2(y)}\right)(f_1(x)g_1(y) + f_2(x)g_2(y)) \\
\left(\frac{1}{g_1(y)}\left(\frac{f_1(x)}{f_2(x)}\right) + k_2\frac{1}{g_2(y)}\right)y'(x) &= \frac{(f_1(x))^2}{f_2(x)} + k_2 f_1(x)\frac{g_1(y)}{g_2(y)} + f_1(x)\frac{g_2(y)}{g_1(y)} + k_2 f_2(x) \\
\left(\frac{1}{g_1(y)}\left(\frac{f_1(x)}{f_2(x)}\right) + k_2\frac{1}{g_2(y)}\right)y'(x) &= \frac{(f_1(x))^2}{f_2(x)} + f_1(x)\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right) + k_2 f_2(x)
\end{aligned}
$$

Which gives us:

$$
\begin{aligned}
\left(\frac{1}{g_1(y)}\left(\frac{f_1(x)}{f_2(x)}\right) + k_2\frac{1}{g_2(y)}\right)y'(x) - f_1(x)\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right) &= \frac{(f_1(x))^2}{f_2(x)} + k_2 f_2(x) \\
\left(\frac{-1}{k}\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right)'\left(\frac{f_1(x)}{f_2(x)}\right) + k_2\frac{1}{g_2(y)}\right)y'(x) - f_1(x)\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right) &= \frac{(f_1(x))^2}{f_2(x)} + k_2 f_2(x) \\
\left(\frac{-1}{k}\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right)'\left(\frac{f_1(x)}{f_2(x)}\right) + k_2\frac{1}{g_2(y)}\right)y'(x) - \frac{1}{k}\left(\frac{f_1(x)}{f_2(x)}\right)'\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right) &= \frac{(f_1(x))^2}{f_2(x)} + k_2 f_2(x) \\
-\frac{1}{k}\left(\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right)\frac{f_1(x)}{f_2(x)}\right)' + k_2\left(\frac{1}{g_2(x)}\right) &= \frac{(f_1(x))^2}{f_2(x)} + k_2 f_2(x) \\
-\frac{1}{k}\left(k_2\frac{g_1(y)}{g_2(y)} + \frac{g_2(y)}{g_1(y)}\right)\frac{f_1(x)}{f_2(x)} + k_2\int\left(\frac{1}{g_2(y)}\right) &= \int\left(\frac{(f_1(x))^2}{f_2(x)} + k_2 f_2(x)\right)
\end{aligned}
$$

But this is precisely what we wished to show. Notice, we can ignore the case where $k = 0$ as this reduces to a separable differential equation. Also, if $k_2 = 0$ then our condition reduces to a more restrictive case of our previous test. Thus this case can also be ignored. In all of this, we are assuming we never divide by zero, (i.e $f_2(x)$, $g_2(y)$ and $g_1(y)$ are nonzero). Also, we have assumed all of these functions are defined on the region under consideration.

3 Examples

Example 1:
Solve:

$$y'(x) = y(x)f(x) + g(x)$$

Solution:
Let $b_2(y) = y$, and $b_1(y) = 1$. Then $b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = 1$ therefore,

$$e^{-\int f(x)}y(x) = \int e^{-\int f(x)}g(x)$$

$$y(x) = e^{\int f(x)}\int(e^{-\int f(x)}g(x))$$

Example 2:
Solve:

$$y'(x) = e^{ky(x)}f(x) + g(x)$$

Solution:
Let $b_1(y) = e^{ky(x)}$, and $b_2(y) = 1$ (where $k \neq 0$). Then $b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = -k$ therefore,

$$e^{k\int g(x)}e^{-ky(x)} = -k\int f(x)e^{k\int g(x)}$$

$$e^{-ky(x)} = -ke^{-k\int g(x)}\int f(x)e^{k\int g(x)}$$

$$y(x) = -\frac{1}{k}ln\left(-ke^{-k\int g(x)}\int f(x)e^{k\int g(x)}\right)$$

example 3:
Solve:

$$y'(x) = y(x)^n f(x) + y(x)g(x)$$

(Assume $n \neq 1$ as that is a trivial case and corresponds to a separable differential equation).
Let $b_1(y) = y(x)^n$ and $b_2(y) = y(x)$. Then $b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = 1 - n$. therefore,

$$e^{(n-1)\int g(x)}y(x)^{1-n} = (1-n)\int f(x)e^{(n-1)\int g(x)}$$

$$y(x) = \left(e^{(1-n)\int g(x)}(1-n)\int f(x)e^{(n-1)\int g(x)}\right)^{\frac{1}{1-n}}$$

example 4:
Solve

$$y' = y^a e^{ky^{1-a}}f(x) + y^a g(x)$$

Let $b_1(y) = y^a e^{ky^{1-a}}$ and let $b_2(y) = y^a$. Then $b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = (a-1)k$. Therefor,

$$e^{(1-a)k\int g(x)}\frac{y^a}{y^a e^{ky^{1-a}}} = (a-1)k\int f(x)e^{(1-a)k\int g(x)}$$

$$e^{-ky^{1-a}} = e^{(a-1)k\int g(x)}(a-1)k\int f(x)e^{(1-a)k\int g(x)}$$

$$-ky^{1-a} = (a-1)k\int g(x) + ln\left((a-1)k\int f(x)e^{(1-a)k\int g(x)}\right)$$

$$y = \left((1-a)\int g(x) - \frac{1}{k}ln\left((a-1)k\int f(x)e^{(1-a)k\int g(x)}\right)\right)^{a-1}$$

example 5:
Now solve,

$$y' = (a_1 y^a e^{ky^{1-a}} + a_2 y^a)f(x) + (a_3 y^a e^{ky^{1-a}} + a_4 y^a)g(x)$$

Rearranging terms will quickly reduce this problem to the problem of the previous question. However, if we where to simply read the functions of y off, we

would find as they are written, the first test does not work, but the second does. Let $b_1(y) = y^a(a_1 e^{ky^{1-a}} + a_2)$ and let $b_2(y) = y^a(a_3 e^{ky^{1-a}} + a_4)$. Then

$$b_1(y)\left(\frac{b_2(y)}{b_1(y)}\right)' = y^a(a_1 e^{ky^{1-a}} + a_2)\left(\frac{a_3 e^{ky^{1-a}} + a_4}{a_1 e^{ky^{1-a}} + a_2}\right)' \neq p$$

For any constant p. (assuming both $a_3 \neq 0$, and $a_2 \neq 0$). Similarly,

$$b_2(y)\left(\frac{b_1(y)}{b_2(y)}\right)' = y^a(a_3 e^{ky^{1-a}} + a_4)\left(\frac{a_1 e^{ky^{1-a}} + a_2}{a_3 e^{ky^{1-a}} + a_4}\right)' \neq p$$

For any constant p. (assuming both $a_1 \neq 0$, and $a_4 \neq 0$). Notice, if either $a_2 = 0$, and $a_3 = 0$, or $a_1 = 0$, and $a_4 = 0$ then the problem reduces. So assuming neither $a_2 = 0$, and $a_3 = 0$, nor $a_1 = 0$, and $a_4 = 0$. Let us run this equation through our other test

$$k_3 = \frac{\left(\frac{b_2(y)}{b_1(y)}\right)'}{\left(\frac{b_1'(y)b_2(y)}{b_1(y)}\right)' - b_2''(y)}$$

$$k_4 = \frac{\left(\frac{b_1(y)}{b_2(y)}\right)'}{\left(\frac{b_2'(y)b_1(y)}{b_2(y)}\right)' - b_1''(y)}$$

We get:

$$k_3 = \frac{1}{(a-1)ka_4},\ and\ k_4 = \frac{1}{(a-1)ka_2}$$

Notice, we assume $a \neq 0$. If $a = 1$ then the equation in question reduces to a separable equation.Theorem 7 will then give us the solution. This tells us that the test will likely only solve differential equations which can be reordered to be solvable by the test in Theorem 6.